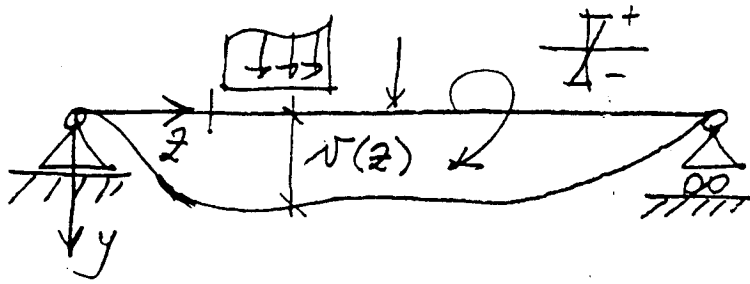


Trave ad asse rettilineo, con assi  $x, y$  principali d'inertzia.



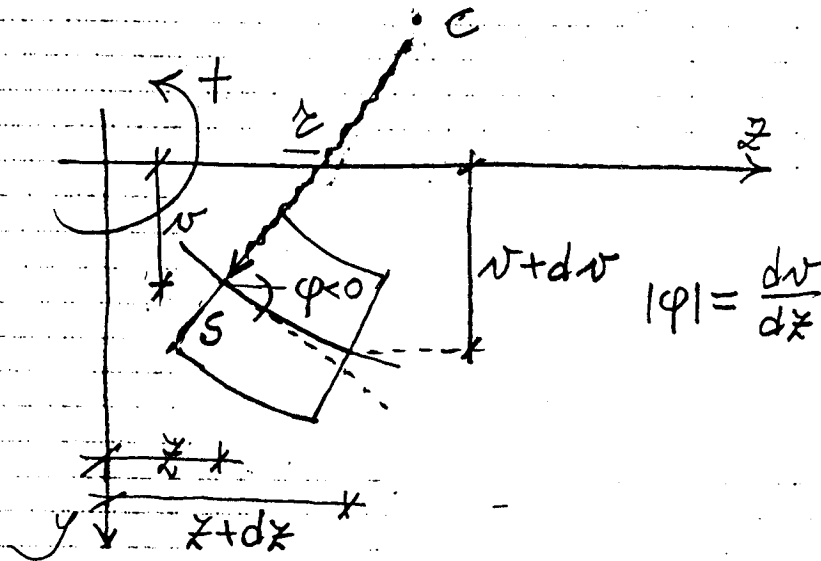
Si assume che la trave sia soggetta a vincoli che impediscono spostamenti  $u(z)$ ,  $w(z)$  e rotazioni  $\varphi(z)$ ,  $\varphi(z)$ , ovvero che le azioni agenti  $y$  non  $z$  inducono questo tipo di spostamenti e rotazioni.

Si vogliono determinare gli spostamenti  $v(z)$  trasversali e le rotazioni  $\varphi(z)$  delle sezioni rette.

Si esaminiamo:

- 1) relazione tra curvatura  $K_x$  e spostamenti  $v(z)$  in assenza di deformazioni da taglio;
- 2) curvatura  $K_x$  di tipo elastico (legata al momento flettente  $M$ ) o indotta da distorsioni termiche;
- 3) esempi -

Ipotesi: le sezioni rette restano piane ed ortogonali all'asse (TRAVERSE DI EULERO-BERNOULLI)



- $C$ : centro di curvatura
- $\rho$ : raggio di curvatura, positivo se  $\rho \cdot y > 0$
- $K_x = \frac{1}{\rho}$ : curvatura (in figura curvatura positiva)

Per piccoli spostamenti:

$$\left[ \varphi \approx \operatorname{tg} \varphi = -\frac{dv}{dx} \right] \quad \begin{array}{l} \text{rotazione della sezione} = \\ = \text{inclinazione della tangente all'asse} \end{array}$$

$$\left[ K_x = \frac{1}{\rho} = \frac{-\frac{d^2v}{dx^2}}{\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{3/2}} \approx -\frac{d^2v}{dx^2} = -v'' \right]$$

o anche:

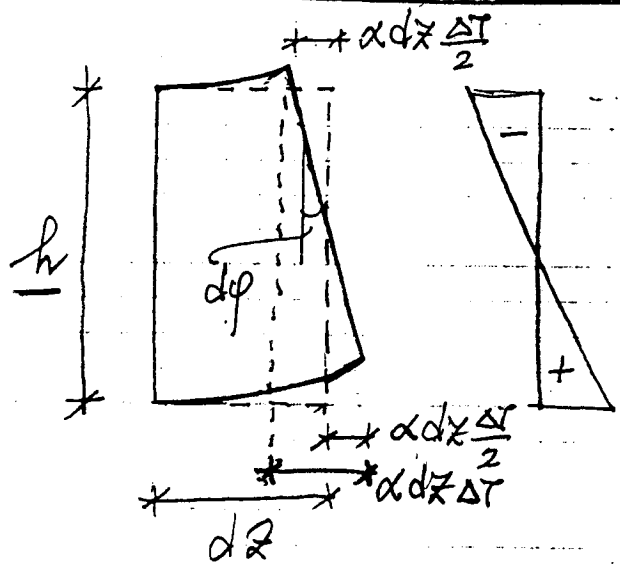
$$K_x = +\frac{d\varphi}{dx} = -\frac{d^2v}{dx^2} \quad \begin{array}{l} \text{è la rotazione relativa} \\ \text{tra due facce di un concio} \\ \text{con } dx = 1 \end{array}$$

$$\boxed{\varphi(z) = -v'(z)}$$

$$\boxed{\kappa(z) = \frac{1}{r(z)} = +\varphi'(z) = -v''(z)}$$

talvolta si usa la notazione  $\chi$  per la curvatura

DISTORSIONE TERMICA "A FARFALLA"



$$-\frac{\Delta T}{2}$$

$$\boxed{\Delta T > 0}$$

$$+\frac{\Delta T}{2}$$

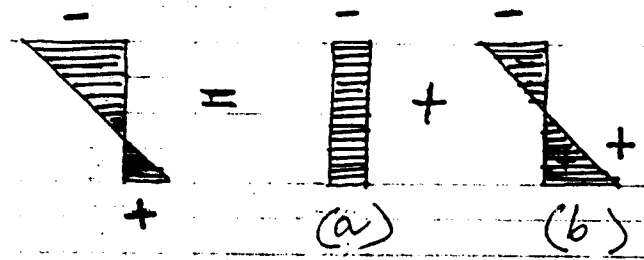
$\alpha$ : coefficiente di dilatazione termica

$$\Rightarrow d\varphi = \frac{\alpha dz \Delta T}{h}$$

$$\Rightarrow \boxed{\kappa = +\frac{d\varphi}{dz} = +\frac{\alpha \cdot \Delta T}{h}}$$

$$\Delta T > 0 \Rightarrow \kappa > 0$$

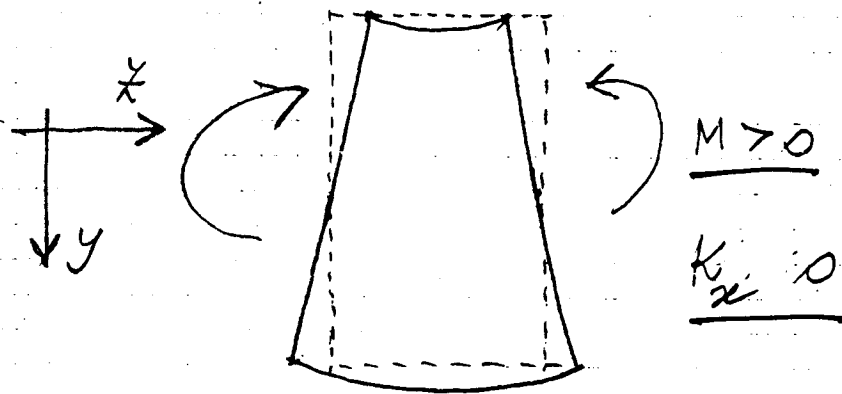
osservazioni:



Solo la (b) induce curvatura

# CURVATURA DOVUTA A MOMENTO FLETTENTE

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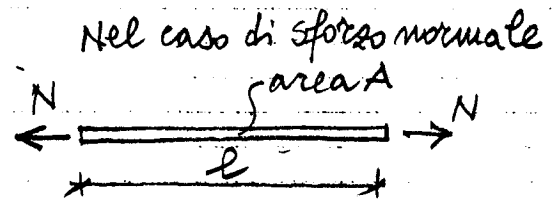


elasticità lineare

$$k_x(x) = C(x) M(x) = \frac{M(x)}{E(x) J(x)}$$

→ [RESTA DA DIMOSTRARE CHE  $C(x) = \frac{1}{E(x) J(x)}$ ]

RIEPILOGO



Nel caso di sforzo normale  
 $\sigma = \frac{N}{A}$  tensione

$\epsilon = \frac{\Delta l}{l}$  deformazione unitaria

legame elastico lineare

$\sigma = E \epsilon$ ; E: modulo di Young

- In presenza di una curvatura  $k_x(x)$ , ed indipendentemente dalla sua origine, risulta, per piccoli spostamenti:

$$v'' = -k_x \quad \rightarrow \quad v' = -\varphi$$

- In presenza di distorsioni termiche "a farfalla"

$$k_x = + \frac{\alpha \Delta T}{h}$$

- In presenza di momento flettente

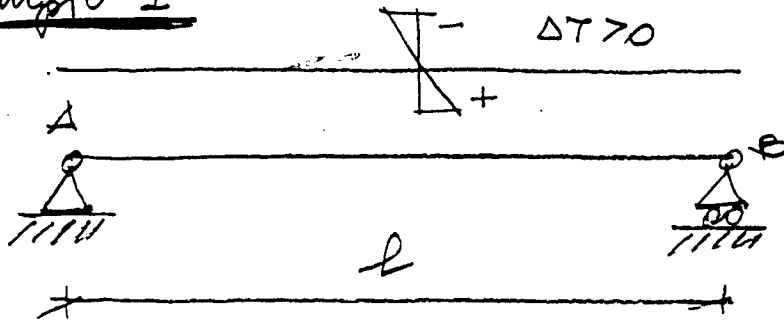
$$k_x = + \frac{M}{EJ}$$

Sovrapposizione degli effetti:

$$k_x = + \frac{M}{EJ} + \frac{\alpha \Delta T}{h}$$

# Esempio 1

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$\alpha$  costante  
 $h$  costante

$$\Delta T > 0 \text{ e costante} \Rightarrow \kappa_x = + \frac{\alpha \Delta T}{h} \text{ costante}$$

$$\Rightarrow \underline{v''} = \left( -\kappa_x = -\frac{\alpha \Delta T}{h} \right) \text{ e integrando:}$$

$$\underline{v'} = -\frac{\alpha \Delta T}{h} z + c$$

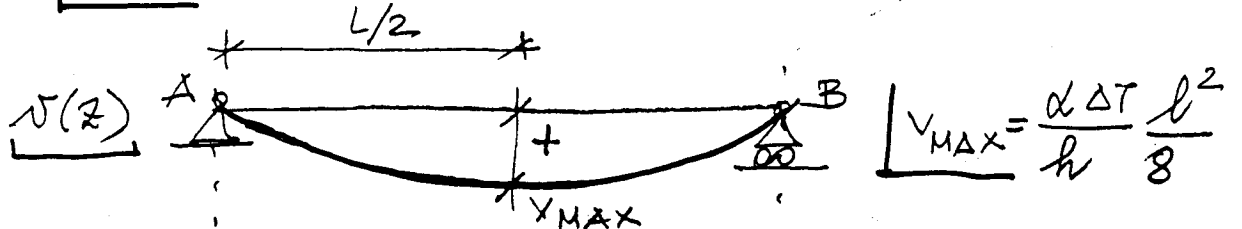
$$\underline{v} = -\frac{\alpha \Delta T}{h} \frac{z^2}{2} + cz + d$$

condizioni al contorno:

$$v_A = v(0) = 0 \Rightarrow d = 0$$

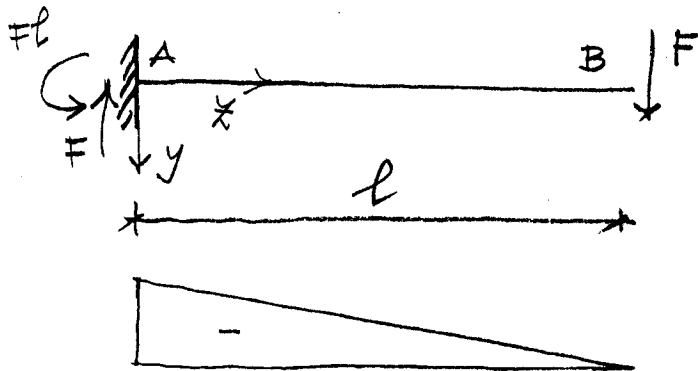
$$v_B = v(l) = 0 \Rightarrow cl = \frac{\alpha \Delta T}{h} \frac{l^2}{2} \Rightarrow c = \frac{\alpha \Delta T}{h} \frac{l}{2}$$

$$\Rightarrow \underline{v(z)} = \frac{\alpha \Delta T}{h} \left( \frac{l}{2} z - \frac{z^2}{2} \right) \text{ parabola}$$



$$\underline{v'(z)} \quad \quad \quad v' = -\varphi = \frac{\alpha \Delta T}{h} \left( \frac{l}{2} - z \right)$$

$$\underline{\varphi_A = -v'(0) = -\frac{\alpha \Delta T}{h} \frac{l}{2}} ; \quad \underline{\varphi_B = -v'(l) = \frac{\alpha \Delta T}{h} \frac{l}{2}}$$

Esempio 2

EY costante

$$M(x) = F(x-l)$$

$$K_x = \frac{M}{EY} = \frac{F(x-l)}{EY} \Rightarrow EY v'' = -M(x) = Fl - Fx$$

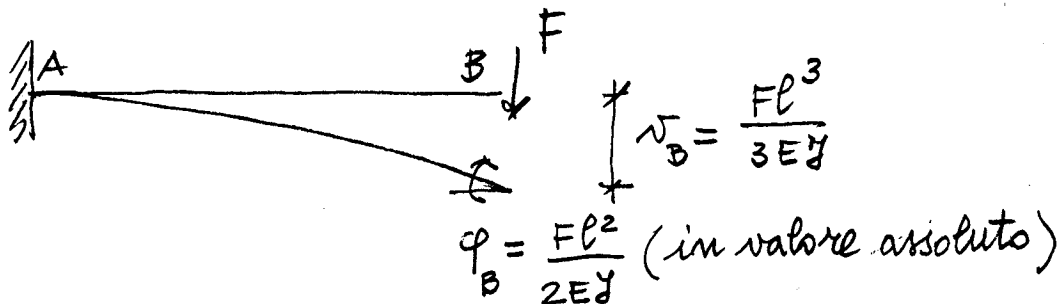
$$\Rightarrow EY v' = Flx - F\frac{x^2}{2} + C$$

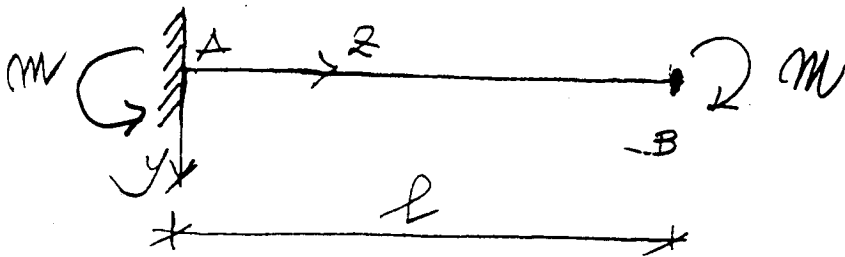
$$EY v = Fl\frac{x^2}{2} - \frac{Fx^3}{6} + Cx + D$$

condizioni al contorno:

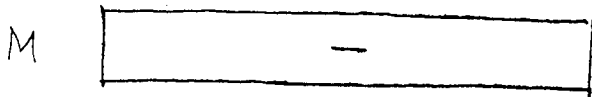
$$v_A = v(0) = 0 \Rightarrow D = 0 \quad ; \quad \varphi_A = -v'_A = -v'(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow v(x) = \frac{F}{EY} \left( \frac{lx^2}{2} - \frac{x^3}{6} \right) \quad ; \quad v'(x) = \frac{F}{EY} \left( lx - \frac{x^2}{2} \right)$$



Esempio 3

EY costante



$$M(x) = -M$$

$$K_x = + \frac{M}{EY} = - \frac{M}{EY}$$

$$\text{ossia } v'' = \frac{M}{EY}$$

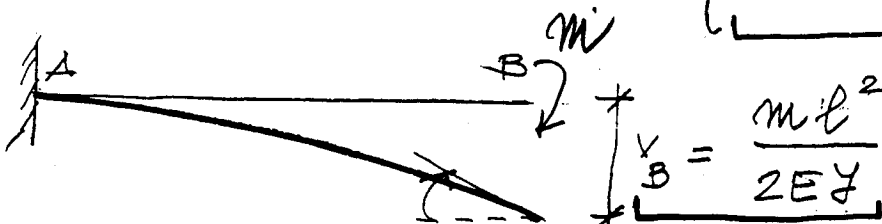
$$v'(x) = \frac{M}{EY} x + C \quad ; \quad v(x) = \frac{M}{EY} \frac{x^2}{2} + Cx + d$$

condizioni al contorno:

$$v_A = v(0) = 0 \Rightarrow d = 0$$

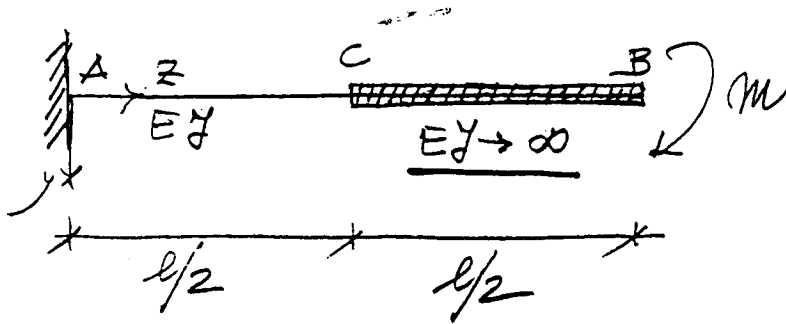
$$v'_A = v'(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow \begin{cases} v(x) = \frac{M}{EY} \frac{x^2}{2} \\ v'(x) = \frac{M}{EY} x \end{cases}$$



$$v_B = \frac{M l^2}{2 E Y}$$

$$\varphi_B = \frac{M l}{E Y} \text{ (in valore assoluto)}$$

Esempio 4

$M(z)$   $M(z) = -M, 0 \leq z \leq l$

$K_x(z)$  
$$\begin{cases} K_x(z) = \frac{-M}{EI}, 0 \leq z < \frac{l}{2} \\ K_x(z) = 0, \frac{l}{2} < z \leq l \end{cases}$$

Tra A e C (v. esempio 2):

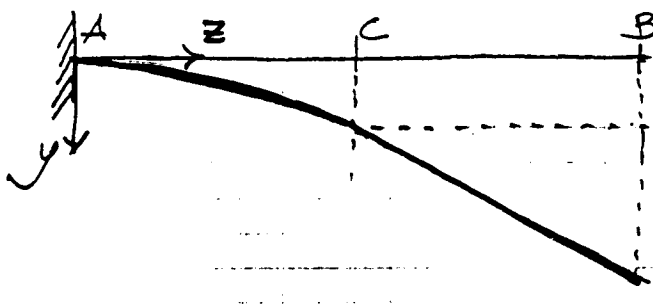
$$\underline{v(z)} = \frac{M}{EI} \frac{z^2}{2}, \quad \underline{v'(z)} = \frac{M}{EI} z, \quad 0 \leq z \leq \frac{l}{2}$$

In C  $\therefore v_C = \frac{M}{EI} \frac{l^2}{8}, \quad v'_C = \frac{M}{EI} \frac{l}{2}$

Tra C e B rotazione rigida:

$$\underline{v'(z)} = v'_C = \frac{M}{EI} \frac{l}{2}$$

$$\underline{v(z)} = v_C + v'_C \left(z - \frac{l}{2}\right) = \frac{M}{EI} \frac{l^2}{8} + \frac{M}{EI} \frac{l}{2} \left(z - \frac{l}{2}\right)$$



$$\boxed{v_B = \frac{3}{8} \frac{Ml^2}{EI}}$$

$$\boxed{v_B \text{ es. 3} = \frac{1}{2} \frac{Ml^2}{EI}}$$